About polynomials of degree 3.

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Find all polynomials $P(x)$ of degree 3 such that
for all negative real numbers x and y

$P(x+y) \geq P(x)+P(y)$.

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Let $P(x) = ax^3 - bx^2 + cx - d$. Then $P(x+y) \geq P(x)+P(y) \Leftrightarrow$

(**1**) $3axy(x+y) - 2bxy + d \geq 0$ for any $x, y \in (-\infty, 0)$.

To find necessary conditions for coefficients a, b and d we set $x = y$ in inequality (**1**) and obtain inequality $6ax^3 - 2bx^2 + d \geq 0$ for any $x < 0$.

Hence, $d \geq \lim\limits_{x\to 0-}(-2bx^2 - 6ax^3) = 0$ and $a \leq \lim\limits_{x\to-\infty-} \frac{1}{6}\left(\frac{2b}{x} - \frac{d}{x^3}\right) = 0$. Thus, $d \geq 0$ and $a < 0$.

Since $2b \leq 6ax + \frac{d}{x^2}$ for any $x < 0$ and by AM-GM Inequality

$6ax + \frac{d}{x^2} = 2 \cdot |3ax| + \frac{d}{x^2} \geq 3\left(9a^2x^2 \cdot \frac{d}{x^2}\right)^{1/3} = 3(9a^2d)^{1/3}$ then $b \leq \frac{3(9a^2d)^{1/3}}{2}$.

To complete the solution we will prove that inequality $3axy(x+y) - 2bxy + d \geq 0$ holds for any $x, y < 0$ if $d \geq 0$, $a < 0$ and $b \leq \frac{3(9a^2d)^{1/3}}{2}$.

We have $3axy(x+y) - 2bxy + d \geq 0 \Leftrightarrow 3a(x+y) + \frac{d}{xy} \geq 2b$ and by AM-GM Inequality

$3a(x+y) + \frac{d}{xy} = 3(-a)(-x + (-y)) + \frac{d}{xy} = 3|a||x| + 3|a||y| + \frac{d}{|x| \cdot |y|} \geq$

$3\left(3|a||x| \cdot 3|a||y| \cdot \frac{d}{|x| \cdot |y|}\right)^{1/3} = 3(9a^2d)^{1/3} \geq 2b$.

Thus, polynomial $P(x) = ax^3 - bx^2 + cx - d$ satisfies $P(x+y) \geq P(x)+P(y)$ for any $x, y < 0$ if and only if $a < 0, d \geq 0, b \leq \frac{3(9a^2d)^{1/3}}{2}$ (c can be any real).